# Research on the Computational Complexity of Walls and Warriors Puzzle

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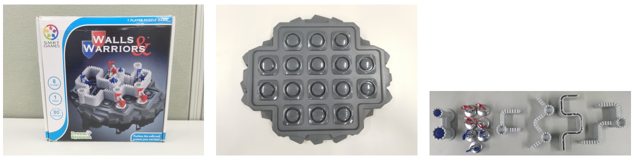
## 1 Introduction

The *Walls & Warriors* (W&W) puzzle [1] is a 2D grid-based logic board game where the player places wall segments to form a single loop making a "castle" area. The configuration must ensure all blue warriors are inside the loop, while all red warriors remain outside, as shown in Fig. 1. This paper explores both its intractability and fixed-parameter tractability.




(a) Game (b) Board (c) Pieces

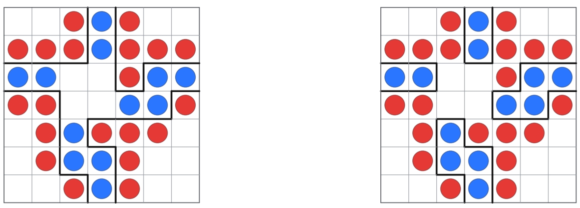
Figure 1: Physical components of the W&W puzzle.

## 2 Mathematical Modeling

We model the board as a grid graph $G = (V, E)$. An instance $\mathcal{I} = (G, B, R, S)$ consists of blue warriors $B$, red warriors $R$, and wall segments $S$ (placed on $G$). A valid configuration $C \subseteq S$ must: (1) induce a single simple cycle; (2) separate units $B \subseteq \text{Int}(C)$ from $R \subseteq \text{Ext}(C)$; (3) ensure $\text{Ext}(C)$ is path-connected to the boundary.

## 3 Intractability

**Theorem 1.** *The generalized 1-loop Walls & Warriors puzzle is NP-complete.*



(a) Unbroken State (b) Broken State

Figure 2: The two unique states of the Vertex Gadget.

**Proof Sketch:** NP-hardness is proved via reduction from Planar 4-regular Tree-Residue Vertex-Breaking (TRVB) [2]. We construct $7 \times 7$ vertex gadgets where a dense arrangement of warriors forces two states: unbroken (connecting incident edge corridors) and broken (severing connections), as shown in Fig. 2. Using edge corridors for rectilinear embedding, a valid TRVB tree residue maps to a hole-free connected interior in W&W. The loop and connectivity constraints ensure the resulting structure is acyclic and connected.

**Theorem 2.** *The 1-loop W&W puzzle is NP-complete even with a perimeter length limit $\ell'$.*

**Proof Sketch:** We reduce from Rectilinear Steiner Tree (RST). An RST of length $\ell$ corresponds to a wall perimeter $\ell' = 6\ell + 4$. By using $3 \times 3$ Terminal and Empty gadgets, red warriors at corners forbid $2 \times 2$ interior blocks, forcing the solution to be a tree polyomino. This establishes a bijection between the RST cost and the W&W perimeter.

## 4 Tractability

**Theorem 3.** *W&W is FPT with respect to board height $H$, solvable in $O(W \cdot H \cdot 8^H)$, where $W$ is the width of the board.*

**Algorithm:** We employ a Plug Dynamic Programming algorithm. The state $S = (P, U, C, Q)$ tracks horizontal plug connectivity ($P$), vertical segments ($U$), region chromatic status ($C$), and loop closure ($Q$). The state space is bounded by the Catalan number $C_{H+1} \approx 4^H$ and a $2^H$ color mask, total $O(8^H)$ states per frontier. The total time complexity is $O(W \cdot H \cdot 8^H)$. Transitions merge components via Union-Find and prune states violating warrior constraints.

| $H$ $(W = 10)$ | States | Time (s) | $W$ $(H = 6)$ | Time (s) |
|---|---|---|---|---|
| 4 | 44 | 0.0011 | 10 | 0.0549 |
| 6 | 7,640 | 0.0702 | 100 | 1.3764 |
| 8 | 115,106 | 0.9807 | 200 | 2.8995 |

Table 1: Plug DP runtime performance on Apple M4.

## 5 Conclusion

We successfully proved the W&W puzzle is NP-complete via two polynomial reductions from TRVB and RST, establishing its general intractability. We also showed it becomes Fixed-Parameter Tractable when grid height is bounded, providing an $O(W \cdot H \cdot 8^H)$ dynamic programming algorithm. This dual nature ensures the puzzle provides a rich mathematical challenge space while remaining solvable for human-playable bounded instances.

## References

[1] SmartGames, "Walls & warriors (online)," https://www.smartgames.eu/uk/one-player-games/walls-warriors, 2026-01-05.

[2] E. D. Demaine and M. Rudoy, "Tree-residue vertex-breaking: A new tool for proving hardness," *arXiv preprint arXiv:1706.07900*, 2018.